Stochastic Frontiers for Six Environmental Condition Indicators: An Examination of the Impacts of Heteroskedasticity

by

Stephen P. Davies
Associate Professor
Colorado State University

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1. Introduction

The use of stochastic frontier cost and production functions has continued unabated over the past years. Empirical applications using panel data and cross sectional models have been made recently and methodological development has also continued. Recent examples include the development of semi-parametric functions for the frontier and development of a non-parametric form for the error term of the stochastic frontier model. Altogether, since the original introduction by Aigner, Lovell and Schmidt (1977) (hereafter ALS), there have been over 100 applications and extensions of the procedure to date.

One aspect of this literature that has been relatively neglected is the role of heteroskedasticity as it affects the performance of a stochastic frontier. For panel data analyses, this is of little concern because the frontier is not specified as part of the error term. However, for models that use a composed error term with a truncated distribution, typically with cross sectional data series, the problem is of considerably greater importance because the variance is correlated with the mean of the error term, and estimation in the presence of heteroskedasticity is biased and inconsistent (Caudill et. al., 1995) (hereafter CFG).

Yet of the large number of applications, only CFG have completely modelled heteroskedasticity in the context of a cross

sectional model of a stochastic frontier. Earlier efforts to incorporate heteroskedasticity included work of Yuengert (1993), and Reifschneider and Stevenson (1991). Yuengert (1991) added a series of group indicators in the variance to capture the effects of firm size, but, as noted by CFG, he only allowed for seven different categories, and so did not deal with firm level inefficiency measures. Reifschneider and Stevenson (1991) added heteroskedasticity to the composite error term by letting the one-sided error have a changing mean using a Normal-Truncated Normal model, rather than capturing the variance directly. The CFG paper developed an approach to model heteroskedasticity directly for the Normal-Half-Normal stochastic frontier and used a frontier cost function to analyze impacts in the banking industry. They demonstrated that estimation without accounting for heteroskedasticity was indeed biased, and also that the coefficients were biased in a particular direction, a fact that was theoretically established by Olson, Waldman and Schmidt (1980). Also, they concluded that any correction is an improvement over ignoring the presence of heteroskedasticity, and that bias would be reduced. The purpose of our paper is to extend these conclusions using a very different and larger data set, and a combination of cost and production function analyses.

In the following sections, we restate the approach of CFG to modeling frontiers with heteroskedasticity, and then we present the empirical results from our analyses of six environmental condition indicators. Our analysis examined national data,

collected from each county in the country, for water quantity (stream flow), water quality (sediment discharge), threatened and endangered animals, the presence of exotic animal species, effects on bird migration patterns, and indexes of the environmental habitat throughout the country. The final section makes several conclusions. In particular, we find support for the CFG contention that there is bias in a particular direction but we also find that non-convexity is a problem with their approach and that it cannot unconditionally be assumed that correcting for heteroskedasticity will improve their assumptions. We also provide suggestions regarding starting values and problems in estimation that were not discussed in their paper.

2. Methodology

The CFG approach begins by recounting the motivation for the stochastic frontier model developed by ALS in their seminal paper. As ALS note, by virtue of using OLS, as long as economists have been examining production functions, they have used average functions rather than a frontier, despite theory's insistence on a frontier. Their suggestion was the stochastic frontier model, presented by CFG as

$$TC_i = X_i'B + W_i + V_i \qquad (1)$$

where TC_i is the total cost of production, X_i is a set of k explanatory variables, and B is a set of kx1 unknown parameters

requiring estimation. The error term is a composed error consisting of W_i and V_i, with W_i being a two-sided error with zero mean and $V(W_i) = \sigma^2_w$. The usual assumption is that it has a Normal distribution. The V_i is a one-sided distribution, with $E(V_i) > 0$, $E(V_i, V_j) = 0$, for all i and j, i=j, and $V(V_i) = \sigma 2_v$. Also, the W_i and V_i are assumed to be uncorrelated. Thus, the composed error term is very often the sum of Half and full Normal errors, which can be expressed as $\epsilon_i = W_i + V_i$. ALS reported the density function of that variable, $f(\epsilon_i)$, originally developed by Weinstein (1964), as

$$f(\epsilon) = (2/\sigma) f^*(\epsilon/\sigma) F^*(\lambda\epsilon/\sigma)_1 \quad -00 < \epsilon < +00_1 \qquad (2)$$

where $\sigma^2 = \sigma^2_w + \sigma^2_v$, $\lambda = \sigma_v/\sigma_w$, and $f^*(.)$ and $F^*(.)$ are the standard Normal density and distribution functions, respectively. This function is typically maximized using the log of the likelihood function. The $E(\epsilon_i) = -(\sqrt{(2/\Pi)})*\sigma_v$, and $V(\epsilon_i) = (w_i) + V(v_i) = ((\Pi-2)/\Pi)*\sigma^2 v + \sigma^2 w$. The expected value is clearly a function of the variance of σ_v, but not of σ_w. Thus, if the heteroskedasticity is found in the two sided error term, the problem is mainly biased standard errors. However, if heteroskedasticity is found in σ_v, with that variance changing across observations in some manner, the mean of the error term will vary and cause biased and inconsistent parameter estimates.

CFG demonstrate that the addition of heteroskedasticity to the model, when contained solely in the σ_{vi}, involves formulating the model as

$$f_i(\epsilon_i) = (2/\sigma_i) f^*(\epsilon_i / \sigma_i) F^*(\lambda_i / \epsilon_i), \quad -00 < \epsilon_i < +00, \tag{3}$$

where $\sigma^2_i = \sigma^2_w + \sigma^2_{vi}$, $\lambda i = \sigma_{vi}/\sigma_w$, and $f^*(.)$ and $F^*(.)$ are the standard Normal density and distribution functions, respectively. This model is implemented by CFG by assuming a multiplicative form of heteroskedasticity for the σ_{vi}, defined by Greene (1995) as

$$\sigma_{vi}=\sigma \exp(Z_i'\gamma) \tag{4}$$

where the Z_i are a set of variables that are thought to drive the changing variance, and the γ are unknown parameters that need to estimated. If the Z_i contain a constant term, the heteroskedastic error term for the one-sided component of the frontier model becomes

$$\sigma_{vi} = \exp(Z_i'\gamma) \tag{5}$$

Additionally, the variance of W_i is specified by CFG as having a multiplicative form as well, leading to

$$\sigma_w = \exp(\theta) \tag{6}$$

Combining the above equations and taking logs leads to the following specification used for estimation

$$\sum_{i=1}^{N} \ln\left(f_i(\epsilon_i)\right) = \frac{N}{2} Ln(2/\pi) - N \ln\left((e^{(2*Z_i\gamma)} + e^{(2*\theta)})^{.5}\right) - \sum_{i=1}^{N} \left((y_i - X_i'\beta)/(e^{(2*Z_i\gamma)} + e^{(2*\theta)})\right) + \sum_{i=1}^{N} \left(\ln(F^*(Y_i - X_i'\beta) * e^{Z_i\gamma} * e^{-\theta}/(e^{2*Z_i\gamma} + e^{2*\theta})^{.5})\right) \tag{7}$$

This equation provides several vantage points on the nature of the model being estimated and the types of starting values that might make sense. First, the γ parameters are all bounded because they always are found as part of an exponent. Depending on the means of the variables used to model heteroskedasticity, the coefficients need to be small or the model will fail to solve due to numeric overflow. Also, the appropriate restricted likelihood function is based on setting all γ'_js to zero, but is still a function of θ, so that σ^2, in the homoskedastic case presented in equation (7), should be

$$\sigma^2 = ((\exp(0) + \exp(\theta))^2 = 1+\exp(2*\theta) \tag{8}$$

and therefore the restricted model used for likelihood ratio tests will be

$$\sum_{i=1}^{N} \ln\left(f_i(\epsilon_i)\right) = \frac{N}{2} Ln(2/\pi) - N \ln\left((1+e^{(2*\theta)})^{.5}\right) - \sum_{i=1}^{N} \left((y_i - X_i'\beta)/(1+e^{(2*\theta)})\right) + \sum_{i=1}^{N} \left(\ln(F^*(Y_i - X_i'\beta) * e^{-\theta}/(1+e^{2*\theta})^{.5})\right) \tag{9}$$

and not the specification given in equation 7.

While this is most likely what was done by CFG, it is not clear from their paper, nor will this latter restricted model yield the same values for the likelihood function as the general homoskedastic case given in equation (3). Thus, the general model is not what is being tested, but rather a specification assuming multiplicative heteroskedasticity. Being consistent in the presence of non-convexity will allow a better chance of getting appropriate likelihood ratio statistics.

A final point is that the cost functions and production functions are easily adapted from this expression. The last term in equation (7) evaluates the Normal cdf, $F^*(.)$, at points related to each observation, before the natural logarithm is taken. This is the model of a cost function. For a production function, the logarithm is taken for $1-F^*(.)$, and the remainder of the likelihood function remains the same.

3. Data

As is common in a modeling effort such as this, the availability of national-level data largely dictates the specific variables that can be included. The following dependent variables (Z_i) were included as measures of forest and rangeland condition:

SED - Sediment discharge (tons/day)[1]

STR - Streamflow (Cfs)[1]

ANI - Threatened and endangered animals (#/acre)[2]

HAB - Habitat structure index (ratio)[3]

BIR - Native breeding birds (# species)[4]

EXO - Exotic breeding birds (ratio)[4]

The following independent variables ($\tilde{X}$) that account for natural and management inputs were included:

URB - Urban land use/cover (%)[5]

AGR - Agricultural land use/cover (%)[5]

RNG - Rangeland use/cover (%)[5]

FOR - Forest land use/cover (%)[5]

WAT - Water use/cover (%)[5]

WET - Wetlands use/cover (%)[5]

FED - Land area in federal ownership (%)[5]

PVT - Land area in private ownership (%)[5]

EVA - Evaporation (inches/day)[6]

PRC - Rainfall (inches/yr)[6]

NsO - Nsowfall (inches/yr)[6]

TMX - Maximum temperature (F°)[6]

TMN - Minimum temperature (F°)[6]

ELV - Elevation (meters)[7]

ELS - Elevation (standard deviation)[7]

POP - Population (#/acre)[8]

CRP - Land area in conservation reserve program (%)[9]

IRR - Land area irrigated (%)[9]

CRU - Land area unlikely to be converted to cropland (%)[10]

CRM - Land area with medium potential to be converted to cropland (%)[10]

CRH - Land area with high potential to be converted to cropland

(%)[10]

DIV - Economic diversity (index)[11]

The following independent variables ($\tilde{Y}$) that measure intensity of commodity harvest were included:

TBR - Stumpage value ($/acre)[12]

BFC - Beefcows (#/acre)[13]

MIN - Minesites (#/acre)[14]

HWY - Land area in highways and interstates (%)[15]

RDS - Land area in dirt, gravel, and paved local roads (%)[15]

We experimented with a variety of functional forms, but the translog was clearly superior based on R^2 and t-test results (recognizing that R^2 coefficients are not comparable with different dependent variable transformations). We handled zero observations in the log transformations by substituting an arbitrarily small constant (which was one-fourth the size of the smallest observation). With 27 original independent variables, the full translog equations have 405 variables included. Eliminating insignificant variables had a minimal effect on R^2, so the equation estimations reported here include only variables with the t statistics indicating significance at the 95% confidence level. We used a forward stepwise procedure to arrive at these variable selections.

4. Results

Summary results for the six stochastic frontiers that converged are presented in Table 1. The first panel of the table

gives key descriptive results from the initial OLS and homoskedastic regressions. The first four columns give the number of variables, the adjusted R^2 and the results of Breusch-Pagan tests for heteroskedasticity in both OLS and the homoskedastic frontier models. The Adjusted R^2s are remarkably good for a cross section of this size except for the SED and ANI regressions. The tests for heteroskedasticity indicate its presence in all regressions using both OLS and the homoskedastic frontier technique.

The fifth column of the first panel in Table 1 shows the skewness statistics for each of the six regressions. These are the standardized third moments. As shown by Waldman (1982), a positive skewness in the OLS result suggests that more of the probability mass is below zero and the third moment will therefore have a negative coefficient, as will the skewness coefficient that is based on the third moment. This is consistent with a production function, in which inefficiency modelled in the error term is negative relative to the frontier. A negative skewness in the OLS residual will therefore indicate the presence of a cost function.

Table 1

Selected Statistical Tests on Stochastic Frontier Regressions
for Six Environmental Condition Indicators

Regression	# of Variables[1]	Adj. R^2	Breusch-Pagan Tests[2]		Skew-ness[3]	DP Test Value[4]
			OLS	Frontier		
SED	74	.433	1560.6*	1945.9*	0.7777	3.916*
STR	75	.904	606.2*	587.9*	0.8240	4.103*
ANI	25	.291	169.1*	134.3*	-0.1744	-0.978
HAB	93	.948	3022.1*	4832.1*	-1.2653	-5.655*
BIR	71	.798	783.9*	783.8*	-0.1831	-1.026
EXO	52	.777	260.7*	221.3*	-0.3244	-1.788**

Notes: [1]With 27 original independent variables, the full translog equations have 405 variables included. The number of variables reported here are the results of a forward stepwise procedure retaining variables only the 95% confidence level.

[2]Breusch-Pagan test results for heteroskedasticity using OLS and Homoskedastic frontier residuals. See Greene (1995).

[3]Sk = $\sqrt{b1}$ = m3/(m2^(3/2)), where m2 and m3 are the second and third moments of the OLS residuals, respectively. See Schmidt and Lin (1984).

[4]The statistic DP = $\delta * \sinh^{-1}(Sk/\lambda)$ = $\delta*(.4343*\log(Sk*(1/\lambda)+\sqrt{(Sk*(1/\lambda))^2+1}))$. See D'Agostino and Pearson (1973). The values of δ and $(1/\lambda)$ are 10.738 and 1.2111, which corresponds to tabulations in Table 4 of D'Agostino and Pearson (1973), for 1000 observations, their largest tabulation.

* = Significance at α = .01.
** = Significance at α = .05.

Table 2

Log-Likelihood Values and Likelihood Ratio Tests
Comparing Alternative Stochastic Frontier Regressions
for Six Environmental Condition Indicators

Regression	Log of the Likelihood Values			Likelihood Ratio Tests[1]	
	Homoskedastic Frontier (1)	Het. Dummy Variable Frontier (2)	Het. Translog Frontier (3)	(2) vs. (1)	(3) vs. (1)
SED	-1871.0	-2008.2	-1864.9	-274.5[f]	12.1
STR	-265.2	-48.1	186.2	434.2*	902.7*
ANI	-6556.6	-6743.9	-6512.2	-374.5[f]	88.7*
HAB	66.9	72.9	--	11.8	--
BIR	3268.3	3221.1	3296.7	-94.5[f]	56.7
EXO	-1210.7	-1421.4	-1382.1	-421.5[f]	-343.0[f]

Notes: [calculated] as $LR = 2*(Log(L_u) - Log(L_r))$, which is distributed as a χ_p distribution if the restriction stated in H_o is true.

[f] Denotes that the LR test was invalid.

-- = Regression for the Heteroskedastic Translog Model failed to converge, and the associated test could not be calculated.

* = Significance at $\alpha = .01$.

The results in column five show that the first two regressions are production functions, while the remaining four are cost functions. Understanding the rationale for this outcome is an intriguing effort, it is beyond the scope of this paper. The final column in Table 1 shows the standard Normal value for a function of the skewness coefficient. The statistic is developed by D'Agostino and Pearson (1973); in our case, four of the six regressions showed significant skewness values. Only the ANI and BIR tests would not be rejected with an $\alpha = .05$.

The second Table contains the values of the log-likelihood functions and results of LR tests between the different forms of the model that were estimated. The specific tests compared the homoskedastic version of each frontier, as the restricted case, with heteroskedasticity assumed to be driven by the same modified translog specification used in the base function, and a separate model using variables not included in the original function, but which were used only to model the behavior of the changing variance. These were dummy variables related to the thirteen ecological regions included in the national analysis.

Table 2 indicates a wide range of outcomes across different models using the various estimation techniques. It should be the case that unrestricted forms, such as the translog model of heteroskedasticity or the dummy variable model, should preform better than a restricted model. As this did not happen, we have to question our approach to choosing starting values, which was to use OLS on the squared and standardized OLS residuals as

starting values. Perhaps a better approach would be to use the squared normalized values of the inefficiency term. Also, zero starting values for the γ parameters should always be used as one alternative to show whether any nearby optima are better than the homoskedastic case.

The results for the LR tests were generally clear, with p-values showing clear acceptances or rejections of heteroskedasticity. In four out of six cases, the heteroskedasticity dummy model, very likely an incorrect measure of heteroskedasticity, failed to have a valid LR test result. In contrast the heteroskedastic translog model had valid tests results for all models expect EXO, and indicated the presence of heteroskedasticity in two of four cases. (It may be that our use of the abbreviated translog model leads a White's type approximation to an unknown form of heteroskedasticity, and therefore may be useful in many circumstances when the researcher is unsure of the correct form).

The other clear suggestion is that there may be non-convexity in the model. This is issue is addressed in the last part of the results section. In the immediately following section, the implications of these tests for the choice of model are examined. Because parameter values are twisted with estimations including heteroskedasticity, the outcome of these tests and the choice of form for the heteroskedasticity is extremely important with regard to the final choice of parameter estimates.

Comparing parameter estimates from four alternative estimators of environmental condition indicators. In the course of this project, four groups of estimates were made on six environmental condition indicators: an OLS model was first run; then a homoskedastic stochastic frontier model was estimated; and two forms of heteroskedastic stochastic frontiers were modelled, one using the same set of variables that were the independent variables in the heteroskedasticity translog, and the other assumed that the proper form of heteroskedasticity could be related to simple regional dummy variables. In this section, comparisons are made among all four models for each of the condition indicators considered.

The changes in coefficient values between homoskedastic and heteroskedastic models is one of the main concerns of CFG. They determined that, for cost functions, the bias in slope coefficients yielded a twisting in parameter estimates when heteroskedasticity existed and was accounted for. That is, some coefficients became larger while others became smaller. They did not consider different forms of heteroskedasticity and assumed that any correction of heteroskedasticity was an improvement. Moreover, their conclusion was valid only for a cost function, rather than a production function. The main purpose of this section is to revisit those conclusions for our data sets and regressions.

Table 3 shows an overview comparison of the different models. The number of instances in which coefficients were over

or under stated relative the coefficients found in the heteroskedasticity translog model are given for the five regressions in which all models successively solved. Also, the number of sign changes between the heteroskedastic translog and each of the other models is given. In general the conclusion of CFG is substantiated, in that a substantial proportion of coefficients are lower with the heteroskedastic translog model and others are higher. The exact proportions that increased or decreased varied by regression and the models being compared, but in each case there were significant proportions that varied in each direction, showing the twisting of coefficients found by CFG. Also, the same outcome occurs production frontiers.

Of course, which outcome is appropriate depends on the results of the testing procedure reported in Table 2, so that, in some cases, tests indicate that heteroskedasticity does not exist, or is only of one form and not the other. Accounting for heteroskedasticity when it does not exist would in fact bias the results to the same degree, but in the opposite direction than when it is not corrected for but exists.

The second conclusion drawn from these results is the performance of the dummy variable model of heteroskedasticity relative to OLS and the homoskedastic models. In all cases, the heteroskedasticity dummy variable model has the most different pattern of coefficients when contrasted with the translog heteroskedasticity model. For example, in the SED models, OLS estimation led to 8 coefficients that were greater than the

translog heteroskedasticity model, 10 that were less, and there wee six sign changes. When that model was estimated using the homoskedastic stochastic frontier model, there were 10 greater than and 8 less than, with only three sign changes relative to the heteroskedasticity translog estimation. Thus, the twisting of the coefficients seemed to progress using the homoskedastic stochastic frontier approach. The heteroskedastic dummy variable model carried this trend even further, in that 12 were over estimated and only 4 were under estimated. The latter model thus seems to differ the most from the heteroskedastic translog model.

Given that the heteroskedasticity dummy variable version often failed to yield valid LR tests in Table 2, it is possible that the correcting for heteroskedasticity in this manner moves the estimates farther away from their true values, and so qualifies CFG's notion that it is always an improvement to correct for heteroskedasticity. A similar result is portrayed in the regression for BIR. The LR test failed for the heteroskedastic dummy variable model while those results are most different from the other estimation procedures. The direction in this case is toward under rather than overestimated coefficients.

In addition to the presentation above, it is useful to examine the actual magnitudes of change in the coefficient estimates. This is somewhat problematic for our regressions, as there are a large number of coefficients to be compared. As our main interest is in how the overall impacts of a given variable change with different estimation techniques, we summarize and

compare the responses in terms of elasticities in Tables 4 through 8, for the five models in which all models solved. In each table, the first four columns present the elasticities from each of the four regression techniques. The last three columns compare the percentage differences between elasticities relative to the heteroskedastic translog outcomes. Thus, values of 100 percent points to an equality of the coefficients, values greater than that indicate the a given model's elasticity is higher than the heteroskedastic translog. A negative value equals a sign change.

The elasticities varied from very small values to a remarkable and yet believable 10.231 for the PRT variable in the STR regression, which summarizes the response of stream flow quantities to precipitation increases. In general, however, the orders of magnitude remained similar within a regression across the different estimation techniques. Thus large values, near one, tended to stay there regardless of estimation technique, as did small elasticities. Despite this tendency, there were many percentage changes that were important, and would lead to different levels of concern in policy analysis. As an example, the contribution of agriculture (AGR) to threatened and endangered animals (ANI), a topic of much concern currently, is fairly large and negative, at -.290, when estimated by OLS. This suggests that agriculture reduces, in a fairly significant way, problems of environmental effects on wildlife. However, the heteroskedasticity translog model yields a coefficient of +.023,

which indicates that agriculture in fact adds to problems of endangered wildlife, albeit in a much less serious way than some observers claim. Thus estimation technique can be very important.

There were also several instances of dramatic changes in elasticities with alternative estimation techniques. The elasticities of precipitation (PRT) and evaporation (EVA) on sediment levels (SED) were cut by a third, from very close to unitary elasticities with OLS to quite inelastic using the heteroskedastic translog model. Also, the impact of EVA on endangered animal species (ANI) goes from 2.2 in the OLS model to nearly zero in the heteroskedastic translog.

Table 3

A comparison of elasticities from alternative stochastic frontier models for five environmental condition indicators

(# of Instances)

Coefficient Variation Relative to Het. Translog:	Model Compared: OLS	Homo. frontier	Het - Dum
	..... SED		
GT Het. Translog	8	10	12
LT Het. Translog	10	11	8
Sign Chg.	6	3	4
	..... STR		
GT Het. Translog	14	14	16
LT Het. Translog	7	7	6
Sign Chg.	4	4	3
	..... ANI		
GT Het. Translog	11	11	10
LT Het. Translog	0	3	6
Sign Chg.	10	7	5
	..... BIR		
GT Het. Translog	12	12	10
LT Het. Translog	7	6	9
Sign Chg.	5	6	5
	..... EXO		
GT Het. Translog	12	11	11
LT Het. Translog	10	11	11
Sign Chg.	0	0	0

Table 4

A Comparison of the SED Model Elasticities

Variable	..... Model Elasticities				: .... Models Compared		
	(1)	(2)	(3)	(4)	: (5)	(6)	(7)
URB	-0.113	-0.056	-0.051	0.006	: -1865.0	-918.5	-837.0
AGR	0.003	-0.004	0.008	-0.048	: -6.7	7.5	-16.6
RNG	0.007	0.003	-0.002	0.001	: 672.5	250.5	-141.3
FOR	-0.002	0.011	0.011	0.023	: -8.3	46.8	47.5
WAT	-0.010	-0.006	-0.008	-0.011	: 97.5	57.3	75.7
WET	-0.003	-0.007	-0.006	-0.004	: 63.5	169.2	154.6
FED	0.000	0.001	0.001	0.003	: 4.9	50.2	18.1
PVT	-0.133	-0.050	-0.057	-0.035	: 383.5	144.3	165.8
EVA	0.619	0.558	0.474	0.282	: 219.2	197.6	167.9
PRC	0.917	0.655	0.648	0.345	: 266.0	189.9	187.9
SNO	-0.013	-0.057	-0.048	-0.115	: 11.2	49.7	41.8
TMN	0.436	0.363	0.437	0.255	: 170.8	142.5	171.4
ELV	-0.004	0.035	0.073	0.122	: -2.9	29.0	59.8
ELS	-0.013	-0.050	-0.054	-0.106	: 12.7	47.6	51.3
POP	0.045	0.020	0.036	0.005	: 858.5	382.2	674.8
CRP	0.015	0.023	0.023	0.024	: 63.9	94.6	93.9
IRR	-0.021	-0.030	-0.036	-0.035	: 59.7	86.2	102.3
CRU	-0.018	-0.018	-0.021	-0.020	: 92.7	92.6	106.0
CRM	-0.020	-0.015	-0.017	-0.005	: 384.2	291.4	338.7
CRH	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
DIV	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
TBR	0.022	0.014	-0.002	-0.002	: -991.9	-638.9	105.9
BFC	0.097	0.066	0.068	0.058	: 166.3	113.2	115.8
MIN	0.012	0.010	0.008	-0.002	: -645.9	-562.8	-411.5
HWY	0.008	0.014	0.020	0.024	: 33.5	58.9	81.1
RDS	-0.020	-0.033	-0.035	-0.032	: 63.0	103.6	107.8

Notes: Columns are: (1) - OLS County model elasticities; (2) - Homoskedastic frontier model elasticities; (3) - Heteroskedastic dummy variable model elasticities; (4) - Heteroskedastic Translog model elasticities; (5) - % County elasticity/Het. Translog elasticity; (6) - % Homoskedastic Frontier Elasticity/Heteroskedastic Translog Elasticity; (7) - % Heteroskedastic Dummy Variable elasticity/Heteroskedastic Translog Elasticity

Table 5

A Comparison of the STR Model Elasticities

Variable	..... Model Elasticities				: .... Models Compared		
	(1)	(2)	(3)	(4)	: (5)	(6)	(7)
URB	-0.011	-0.009	-0.021	-0.018	: 59.5	48.9	116.4
AGR	0.007	0.014	0.018	0.019	: 35.6	71.2	94.5
RNG	-0.003	-0.005	-0.013	-0.031	: 8.5	16.7	42.3
FOR	0.052	0.056	0.048	0.037	: 141.3	152.1	130.8
WAT	0.006	0.006	0.002	0.000	: 1365.1	1444.2	423.3
WET	-0.003	-0.004	-0.001	0.001	: -449.1	-617.5	-101.8
FED	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
PVT	-0.037	-0.034	-0.047	-0.041	: 90.6	81.8	113.3
EVA	-0.787	-0.789	-1.041	-0.992	: 79.4	79.6	105.0
PRC	10.231	10.192	10.044	9.922	: 103.1	102.7	101.2
SNO	0.110	0.100	0.061	0.034	: 328.6	299.4	180.5
TMX	-0.457	-0.466	-0.378	-0.321	: 142.6	145.2	118.0
TMN	-0.181	-0.203	-0.206	-0.189	: 95.4	107.4	108.6
ELV	-0.028	-0.019	0.019	0.031	: -89.2	-60.6	61.1
ELS	0.073	0.065	0.039	0.054	: 135.5	119.6	71.9
POP	0.042	0.039	0.051	0.040	: 105.0	95.5	126.6
CRP	-0.005	-0.005	-0.002	0.000	: -1173.2	-1185.4	-375.6
IRR	0.035	0.032	0.014	0.009	: 372.9	348.4	154.5
CRU	-0.017	-0.018	-0.007	-0.005	: 355.3	372.3	137.8
CRM	0.010	0.009	0.006	0.000	: 4096.0	3780.0	2380.0
CRH	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
DIV	-0.174	-0.180	-0.125	-0.106	: 164.0	169.5	118.3
TBR	0.012	0.014	0.010	0.010	: 120.5	145.2	105.6
BFC	-0.017	-0.023	-0.033	-0.051	: 33.5	45.0	63.3
MIN	0.010	0.011	0.002	-0.001	: -1585.9	-1660.9	-332.8
HWY	-0.023	-0.021	-0.006	-0.011	: 211.0	192.3	54.2
RDS	-0.093	-0.091	-0.077	-0.054	: 171.9	167.6	141.9

Notes: Columns are: (1) - OLS County model elasticities; (2) - Homoskedastic frontier model elasticities; (3) - Heteroskedastic dummy variable model elasticities; (4) - Heteroskedastic Translog model elasticities; (5) - % County elasticity/Het. Translog elasticity; (6) - % Homoskedastic Frontier Elasticity/Heteroskedastic Translog Elasticity; (7) - % Heteroskedastic Dummy Variable elasticity/Heteroskedastic Translog Elasticity

Table 6

A Comparison of the ANI Model Elasticities

Variable	..... Model Elasticities				: .... Models Compared		
	(1)	(2)	(3)	(4)	: (5)	(6)	(7)
URB	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
AGR	-0.290	-0.086	-0.006	0.023	: -1297.5	-130.0	-153.3
RNG	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
FOR	0.200	0.087	0.086	0.061	: -1250.0	-369.7	-25.3
WAT	0.296	0.112	0.099	0.031	: ERR	ERR	ERR
WET	0.083	0.002	-0.012	-0.025	: 326.2	141.1	140.1
FED	0.202	0.082	0.073	0.032	: 967.7	365.0	323.3
PVT	0.633	0.072	0.120	-0.088	: -332.2	-6.6	49.5
EVA	2.236	0.750	0.509	0.076	: 621.1	251.4	223.9
PRC	0.974	0.623	0.512	0.319	: -716.0	-81.9	-135.2
SNO	0.121	-0.059	-0.066	-0.149	: 2948.3	988.4	671.4
TMX	2.286	0.032	-0.136	-1.157	: 305.3	195.2	160.4
TMN	-0.733	-0.718	-0.749	-0.646	: -81.2	39.4	44.0
ELV	-0.397	-0.211	-0.250	-0.193	: -197.6	-2.7	11.8
ELS	0.473	0.185	0.118	0.063	: 113.5	111.2	116.1
POP	-0.149	0.093	0.093	0.211	: 205.9	109.7	129.6
CRP	0.000	0.000	0.000	0.000	: 754.1	294.4	187.6
IRR	0.106	0.042	0.009	0.021	: -70.8	44.1	44.2
CRU	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
CRM	0.000	0.000	0.000	0.000	: 494.9	194.5	40.5
CRH	0.043	0.017	0.008	-0.001	: ERR	ERR	ERR
DIV	1.353	-0.744	-0.829	-1.741	: ERR	ERR	ERR
TBR	-0.025	-0.021	-0.022	-0.019	: -3589.2	-1406.7	-679.2
BFC	0.373	0.239	0.208	0.199	: -77.7	42.7	47.6
MIN	0.050	0.019	0.009	-0.001	: 129.2	108.3	114.8
HWY	0.000	0.000	0.000	0.000	: 187.7	120.0	104.6
RDS	0.194	0.019	0.023	-0.015	: -3577.0	-1402.2	-677.0

Notes: Columns are: (1) - OLS County model elasticities; (2) - Homoskedastic frontier model elasticities; (3) - Heteroskedastic dummy variable model elasticities; (4) - Heteroskedastic Translog model elasticities; (5) - % County elasticity/Het. Translog elasticity; (6) - % Homoskedastic Frontier Elasticity/Heteroskedastic Translog Elasticity; (7) - % Heteroskedastic Dummy Variable elasticity/Heteroskedastic Translog Elasticity

Table 7

A Comparison of the BIR Model Elasticities

Variable	..... Model Elasticities				: .... Models Compared		
	(1)	(2)	(3)	(4)	: (5)	(6)	(7)
URB	-0.006	-0.006	-0.003	-0.003	: 190.2	201.7	105.1
AGR	-0.007	-0.008	-0.011	0.001	: -680.0	-694.5	-1030.0
RNG	-0.001	-0.001	0.001	0.002	: -58.3	-43.9	26.0
FOR	0.025	0.025	0.021	0.024	: 105.6	104.4	88.4
WAT	0.006	0.006	0.006	0.006	: 100.5	103.2	106.6
WET	0.005	0.005	0.004	0.005	: 91.6	93.1	80.7
FED	0.000	0.000	0.000	0.001	: -22.2	-11.1	-2.2
PVT	0.001	0.000	0.006	0.002	: 58.4	5.1	269.6
EVA	-0.120	-0.114	-0.046	0.037	: -327.4	-311.2	-126.7
PRC	0.096	0.094	0.036	0.105	: 91.8	89.9	34.1
SNO	0.015	0.015	0.014	0.009	: 161.5	157.7	147.7
TMX	-0.047	-0.054	-0.066	-0.140	: 33.5	38.8	47.3
TMN	-0.196	-0.193	-0.164	-0.178	: 110.4	108.6	92.5
ELV	-0.045	-0.044	-0.042	-0.047	: 95.7	94.6	88.7
ELS	0.006	0.005	0.007	0.004	: 169.9	147.5	183.3
POP	0.004	0.005	-0.001	0.002	: 180.6	218.9	-26.3
CRP	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
IRR	0.000	0.000	-0.001	-0.002	: 1.5	-2.0	35.6
CRU	0.014	0.013	0.010	0.001	: 2419.6	2283.9	1744.6
CRM	0.009	0.009	0.009	0.005	: 169.9	166.4	166.8
CRH	0.002	0.002	0.002	0.001	: 244.9	244.9	196.2
DIV	-0.032	-0.032	-0.025	-0.014	: 223.4	223.8	176.4
TBR	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
BFC	0.003	0.003	0.010	-0.002	: -156.3	-147.7	-496.4
MIN	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
HWY	0.011	0.012	0.012	0.011	: 101.1	109.7	106.5
RDS	-0.011	-0.011	-0.008	-0.012	: 93.0	93.7	69.5

Notes: Columns are: (1) - OLS County model elasticities; (2) - Homoskedastic frontier model elasticities; (3) - Heteroskedastic dummy variable model elasticities; (4) - Heteroskedastic Translog model elasticities; (5) - % County elasticity/Het. Translog elasticity; (6) - % Homoskedastic Frontier Elasticity/Heteroskedastic Translog Elasticity; (7) - % Heteroskedastic Dummy Variable elasticity/Heteroskedastic Translog Elasticity

Table 8

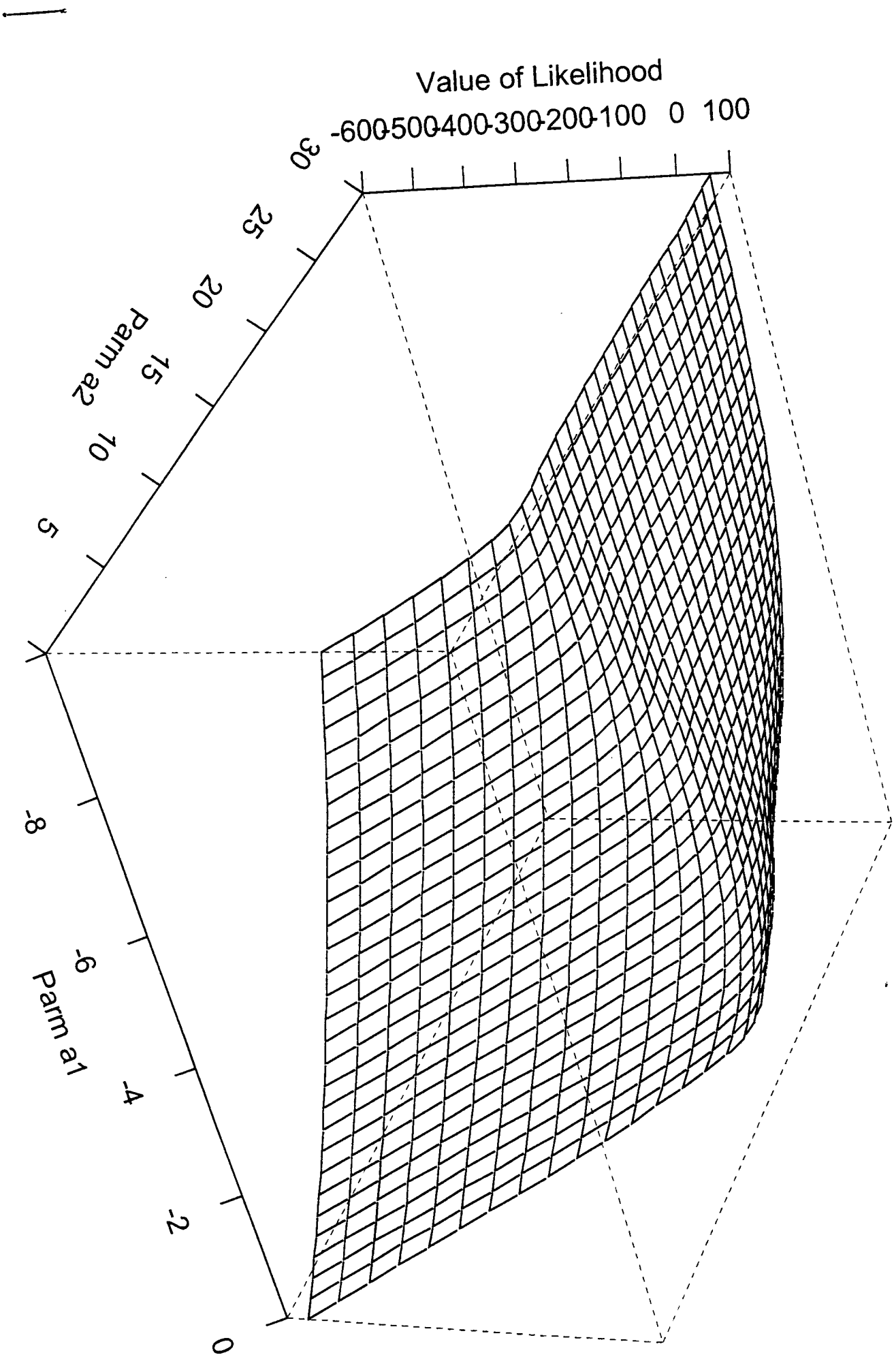
A Comparison of the EXO Model Elasticities

Variable	..... Model Elasticities				: .... Models Compared		
	(1)	(2)	(3)	(4)	: (5)	(6)	(7)
URB	0.089	0.085	0.077	0.073	: 122.2	117.3	105.6
AGR	0.204	0.200	0.199	0.252	: 81.0	79.7	79.0
RNG	-0.020	-0.020	-0.016	-0.015	: 132.5	131.3	105.4
FOR	-0.076	-0.074	-0.072	-0.024	: 320.3	310.6	303.0
WAT	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
WET	-0.018	-0.018	-0.017	-0.015	: 114.9	115.5	112.5
FED	-0.005	-0.006	-0.005	-0.008	: 69.1	77.1	60.4
PVT	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
EVA	0.540	0.614	0.528	1.070	: 50.5	57.4	49.4
PRC	-0.103	-0.096	-0.116	-0.095	: 108.4	101.4	122.3
SNO	0.100	0.098	0.089	0.077	: 130.7	127.4	116.2
TMX	-0.758	-0.756	-0.655	-0.451	: 168.1	167.6	145.3
TMN	-0.030	-0.057	-0.046	-0.467	: 6.5	12.3	9.9
ELV	-0.068	-0.074	-0.082	-0.096	: 70.8	76.8	86.1
ELS	-0.042	-0.038	-0.023	-0.028	: 148.1	133.3	82.9
POP	0.018	0.020	0.016	0.026	: 69.1	77.0	60.4
CRP	0.018	0.018	0.018	0.018	: 100.3	99.9	101.0
IRR	-0.003	-0.004	0.000	-0.011	: 30.9	37.6	3.2
CRU	-0.025	-0.025	-0.026	-0.024	: 106.3	105.3	109.8
CRM	-0.011	-0.012	-0.015	-0.028	: 40.3	43.0	53.0
CRH	0.005	0.005	0.005	0.003	: 175.0	181.9	172.2
DIV	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
TBR	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
BFC	-0.030	-0.033	-0.045	-0.073	: 40.8	45.0	60.7
MIN	0.000	0.000	0.000	0.000	: ERR	ERR	ERR
HWY	0.019	0.022	0.018	0.029	: 66.3	76.8	63.9
RDS	-0.058	-0.055	-0.048	-0.046	: 126.0	120.0	104.5

Notes: Columns are: (1) - OLS County model elasticities; (2) - Homoskedastic frontier model elasticities; (3) - Heteroskedastic dummy variable model elasticities; (4) - Heteroskedastic Translog model elasticities; (5) - % County elasticity/Het. Translog elasticity; (6) - % Homoskedastic Frontier Elasticity/Heteroskedastic Translog Elasticity; (7) - % Heteroskedastic Dummy Variable elasticity/Heteroskedastic Translog Elasticity

Non-convexity of the heteroskedastic stochastic frontier estimator. The final perspective on heteroskedasticity and the estimation of production and cost functions using stochastic frontiers has to do with the possible non-convexity when heteroskedasticity is introduced in the manner proposed by CFG. This possibility was suggested in Table 2, where less restricted likelihood functions yielded poorer log likelihood values. There may be two reasons for this outcome. The first is that the comparison between the restricted and unrestricted models is not done correctly, as we pointed out in the methodology section. The appropriate restricted model is given in equation (8), and is not the same as the typical homoskedastic version given in CFG. The second factor may be that CFG's likelihood function for the heteroskedastic stochastic frontier is not convex.

The graphs in Figure 1 show the likelihood values (on the vertical axis) for different combinations of parameter values for two coefficients in the ANI regression. The axis labelled "Parm a14" is one of the γ parameters, while the "Parm b13" represents one of slope parameters in the ANI regression. Earlier experimentation with a smaller model indicated that most potential problems of non-convexity came with combinations heteroskedasticity terms and coefficients associated with the slope variables. The graph in Figure 1 shows a distinct trough and two separate directions that a gradient search algorithm would follow.


Value of Likelihood
-600 -500 -400 -300 -200 -100 0 100
30
25
20
15
10
5
Parm a2
-8
-6
-4
-2
0
Parm a1

The key value in the figure seems to be when b13 values are greater or less than 3. For values greater than 3, a search algorithm would go to the plateau in the lefthand corner of the Figure, while it would move to the ridge running through the graph at about -5 if starting values are smaller than 3. Values of a14 do not materially affect the outcome of the likelihood function in this particular instance. However, this parameter may be associated with a variable with a small mean and so requires a larger range to see its effects. Typically, the variance should affect the height of the likelihood function and therefore its value. A final observation is that the likelihood surface is very flat across large ranges. As a result, unless tolerances are quite tight, a large range of parameters could be consistent with near optima.

5. Conclusions

This analysis critiques the work of a recent publication by Caudill, Ford and Gropper, who demonstrated the importance of considering heteroskedasticity in frontier estimation. We find that some of their results are consistent across a larger and more diverse data set with different characteristics, but there were several aspects they did not consider that are significant in assessing the impacts of heteroskedasticity on frontier estimation. For example, their empirical example did not indicate skewness or that a homoskedastic frontier existed, while the work here does. Thus, we are able to show the improvements

that corrections for heteroskedasticity bring on while they really could not. Our work also examines the behavior of both production and cost functions, while CFG do not. In particular, we find that the twisting of parameters, so that some coefficients become relatively large and others are small relative to OLS and the homoskedastic case, occurs for both the cost and production functions. Finally, we are concerned with issues of starting values and non-convexity of the estimator, two aspects they did not consider.

CFG did not discuss starting values at all in their paper, and there are several issues there. Given the potential non-convexity of the model, one alternative that needs to be tried is to set the γ parameters to zero to see that the model seeks likelihood values that are improvements over the homoskedastic case. If there are no increases in the likelihood function, the optimization routine will not go downhill in search of a poorer position. We demonstrated that another typical choice of starting values for the γ would be based on an OLS estimation of the hypothesized set of variables against the squared and standardized error terms, where the error terms used are taken from an initial OLS estimation. The assumption here is that the error terms drawn from the OLS estimation vary primarily due to the changing variance of V_i, as the variance of W_i is constant.

This approach may have several limitations. Even if there is not variation in the variance of W_i, the presence of W_i in the V_i creates a scaling problem and will make the estimated γ values

inappropriate for use as starting values. Also, these error terms will certainly be affected by variation in the levels of W_i, and the estimates may be biased for this reason. For these two reasons, it may be important to square and standardize the one-sided error derived from a homoskedastic regression, which has already separated out the effects of the W_i.

The issue of non-convexity was also brought out in the work presented here, which was again an issue not brought up by CFG. We showed graphically that the likelihood surface, at least for one combination of parameters, has a trough and two separate directions that a gradient search would follow. We also showed several cases in Table 2 where a more restricted model led to lower likelihood values, perhaps due to an incorrect comparison between restricted and unrestricted models, but also perhaps due to the non-convexity that appears to exist in the heteroskedasticity likelihood function.

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